2. Koristeći *tabelu 10.1.* osnovnih neodredjenih integrala i njihove osobine, rešiti sledeće integrale:

(a) $\int \left(\sin x + 4e^x - \frac{3}{x}\right) dx$; (b) $\int (3x^3 + 5x^2 - 4x + 7)\, dx$;

(c) $\int \frac{x^4 - x^3 - 8x^2 + x - 2}{x^2}\, dx$; (d) $\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[5]{x^4}} - \sqrt[2005]{x^{2004}}\right) dx$;

(e) $\int \frac{x^2}{1+x^2}\, dx$; (f) $\int \frac{3\sin^2 x + 5}{\cos^2 x}\, dx$; (g) $\int \frac{2^{3x}}{3^{2x}}\, dx$.

Rešenje:

(a)

$$\int \left(\sin x + 4e^x - \frac{3}{x}\right) dx = \int \sin x\, dx + \int 4e^x\, dx - \int \frac{3}{x}\, dx$$

$$= \int \sin x\, dx + 4\int e^x\, dx - 3\int \frac{1}{x}\, dx = -\cos x + 4e^x - 3\ln|x| + C;$$

(b)

$$\int (3x^3 + 5x^2 - 4x + 7)\, dx = \int 3x^3\, dx + \int 5x^2\, dx - \int 4x\, dx + \int 7\, dx$$

$$= 3\int x^3 dx + 5\int x^2 dx - 4\int x\, dx + 7\int dx = \frac{3}{4}x^4 + \frac{5}{3}x^3 - 2x^2 + 7x + C;$$

(c)

$$\int \frac{x^4 - x^3 - 8x^2 + x - 2}{x^2}\, dx = \int x^2\, dx - \int x\, dx - 8\int dx + \int \frac{1}{x}\, dx$$

$$-2\int x^{-2}\, dx = \frac{x^3}{3} - \frac{x^2}{2} - 8x + \ln|x| + \frac{2}{x} + C;$$

(d)

$$\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[5]{x^4}} - \sqrt[2005]{x^{2004}}\right) dx = \int x^{\frac{1}{3}}\, dx + \int x^{-\frac{4}{5}}\, dx - \int x^{\frac{2004}{2005}}\, dx$$

$$= \frac{3}{4}\sqrt[3]{x^4} + 5\sqrt[5]{x} - \frac{2005}{4009}\sqrt[2005]{x^{4009}} + C;$$

(e)

$$\int \frac{x^2}{1+x^2} dx = \int \frac{x^2 + 1 - 1}{1+x^2}\, dx = \int dx - \int \frac{dx}{1+x^2} = x - \operatorname{arctg} x + C;$$

(f)

$$\int \frac{3\sin^2 x+5}{\cos^2 x}\,dx = \int \frac{3(1-\cos^2 x)+5}{\cos^2 x}\,dx = 8\int \frac{1}{\cos^2 x}\,dx - 3\int dx$$

$$= 8\,\mathrm{tg}\, x - 3x + C;$$

(g)

$$\int \frac{2^{3x}}{3^{2x}}\,dx = \int \frac{8^x}{9^x}\,dx = \int \left(\frac{8}{9}\right)^x dx = \frac{1}{\ln\frac{8}{9}}\left(\frac{8}{9}\right)^x + C.$$

3. Koristeći se metodom smene rešiti integrale:

(a) $\int \sin(5x)\,dx$; (b) $\int e^{-4x+7}\,dx$; (c) $\int \frac{3}{7x+8}\,dx$;

(d) $\int \frac{x}{x-1}\,dx$; (e) $\int \frac{6}{(5-x)^2}\,dx$; (f) $\int x\cos(3x^2+2)\,dx$;

(g) $\int (x-1)\,e^{x^2-2x+1}\,dx$; (h) $\int \frac{-x+2}{x^2-4x-7}\,dx$; (i) $\int \frac{1}{x^2+2x+10}\,dx$;

(j) $\int \frac{2x+5}{x^2+2x+10}\,dx$; (k) $\int \frac{x^2+5x-3}{(x-2)^2}\,dx$; (l) $\int \frac{e^x}{e^x-1}\,dx$;

(m) $\int \frac{\sin x-\cos x}{\sin x+\cos x}\,dx$; (n) $\int \frac{\sqrt{2\sqrt{x}+5}}{\sqrt{x}}\,dx$;

(o) $\int \sqrt{x}\sqrt{2\sqrt{x}+5}\,dx$; (p) $\int \frac{\ln x}{x}\,dx$; (q) $\int \frac{\ln^{2005} x}{x}\,dx$;

(r) $\int \frac{1}{x\ln x\ln(\ln x)}\,dx$; (s) $\int \frac{2^{\mathrm{tg}\, x}}{\cos^2 x}\,dx$.

Rešenje:

(a)

$$\int \sin(5x)\,dx = \left[\begin{array}{c} t=5x \\ dt=5\,dx \end{array}\right] = \frac{1}{5}\int \sin t\,dt = -\frac{1}{5}\cos t + C$$

$$= -\frac{1}{5}\cos(5x) + C;$$

(b)

$$\int e^{-4x+7}\,dx = \left[\begin{array}{c} t=-4x+7 \\ dt=-4\,dx \end{array}\right] = -\frac{1}{4}\int e^t\,dt = -\frac{1}{4}e^t + C$$

$$= -\frac{1}{4}e^{-4x+7} + C;$$

(c)

$$\int \frac{3}{7x+8}\,dx = \left[\begin{array}{c} t = 7x+8 \\ dt = 7\,dx \end{array}\right] = \frac{3}{7}\int \frac{1}{t}\,dt = \frac{3}{7}\ln|t| + C$$
$$= \frac{3}{7}\ln|7x+8| + C;$$

(d)

$$\int \frac{x}{x-1}\,dx = \left[\begin{array}{c} t = x-1 \\ x = t+1 \\ dt = dx \end{array}\right] = \int \frac{t+1}{t}\,dt = \int dt + \int \frac{1}{t}\,dt$$
$$= t + \ln|t| + C = x - 1 + \ln|x-1| + C;$$

(e)

$$\int \frac{6}{(5-x)^2}\,dx = \left[\begin{array}{c} t = 5-x \\ dt = -dx \end{array}\right] = -6\int \frac{1}{t^2}\,dt = \frac{6}{t} + C = \frac{6}{5-x} + C;$$

(f)

$$\int x\cos(3x^2+2)\,dx = \left[\begin{array}{c} t = 3x^2+2 \\ dt = 6x\,dx \end{array}\right] = \frac{1}{6}\int \cos t\,dt$$
$$= \frac{1}{6}\sin(3x^2+2) + C;$$

(g)

$$\int (x-1)\,e^{x^2-2x+1}\,dx = \left[\begin{array}{c} t = x^2-2x+1 \\ dt = 2(x-1)\,dx \end{array}\right] = \frac{1}{2}\int e^t\,dt$$
$$= \frac{1}{2}e^{x^2-2x+1} + C;$$

(h)

$$\int \frac{-x+2}{x^2-4x-7}\,dx = \left[\begin{array}{c} t = x^2-4x-7 \\ dt = -2(-x+2)\,dx \end{array}\right] = -\frac{1}{2}\int \frac{1}{t}\,dt$$
$$= -\frac{1}{2}\ln|x^2-4x-7| + C;$$

(i)

$$\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{(x+1)^2+9}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx$$
$$= \left[\begin{array}{c} t = \frac{x+1}{3} \\ dt = \frac{1}{3}dx \end{array}\right] = \frac{1}{3}\int \frac{1}{t^2+1}\,dt = \frac{1}{3}\operatorname{arctg}\left(\frac{x+1}{3}\right) + C;$$

(j)

$$\int \frac{2x+5}{x^2+2x+10}\,dx = \int \frac{2x+2}{x^2+2x+10}\,dx + 3\int \frac{1}{x^2+2x+10}\,dx$$

$$= \begin{bmatrix} t = x^2+2x+10 \\ dt = (2x+2)\,dx \end{bmatrix}, \begin{bmatrix} \text{primer} \\ \text{pod (i)} \end{bmatrix}$$

$$= \ln(x^2+2x+10) + \operatorname{arctg}\left(\frac{x+1}{3}\right) + C;$$

(k)

$$\int \frac{x^2+5x-3}{(x-2)^2}\,dx = \begin{bmatrix} t = x-2 \\ x = t+2 \\ dt = dx \end{bmatrix} = \int \frac{(t+2)^2+5(t+2)-3}{t^2}\,dt$$

$$= \int \frac{t^2+9t+11}{t^2}\,dt = \int dt + 9\int \frac{1}{t}\,dt + 11\int \frac{1}{t^2}\,dt$$

$$= t + 9\ln|t| - 11\,\frac{1}{t} + C = x-2+9\ln|x-2| - \frac{11}{x-2} + C;$$

(l)

$$\int \frac{e^x}{e^x-1}\,dx = \begin{bmatrix} t = e^x-1 \\ dt = e^x\,dx \end{bmatrix} = \int \frac{1}{t}\,dt = \ln|e^x-1| + C;$$

(m)

$$\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \begin{bmatrix} t = \sin x + \cos x \\ dt = -(\sin x - \cos x)\,dx \end{bmatrix} = -\int \frac{1}{t}\,dt$$

$$= -\ln|\sin x + \cos x| + C;$$

(n)

$$\int \frac{\sqrt{2\sqrt{x}+5}}{\sqrt{x}}\,dx = \begin{bmatrix} t = 2\sqrt{x}+5 \\ dt = \frac{1}{\sqrt{x}}\,dx \end{bmatrix} = \int \sqrt{t}\,dt = \frac{2}{3}\,t\sqrt{t} + C$$

$$= \frac{2}{3}\,(2\sqrt{x}+5)\sqrt{2\sqrt{x}+5} + C;$$

(o)

$$\int \sqrt{x}\sqrt{2\sqrt{x}+5}\,dx = \left[\begin{array}{c} t = 2\sqrt{x}+5 \\ \sqrt{x} = \frac{t-5}{2} \\ dt = \frac{1}{\sqrt{x}}dx \end{array}\right] = \int \sqrt{t}\left(\frac{t-5}{2}\right)^2 dt$$

$$= \frac{1}{4}\int (t^{\frac{5}{2}} - 10t^{\frac{3}{2}} + 25t^{\frac{1}{2}})\,dt = \frac{1}{4}\left(\frac{2}{7}t^{\frac{7}{2}} - 4t^{\frac{5}{2}} + \frac{50}{3}t^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{14}(2\sqrt{x}+5)^{\frac{7}{2}} - (2\sqrt{x}+5)^{\frac{5}{2}} + \frac{25}{6}(2\sqrt{x}+5)^{\frac{3}{2}} + C;$$

(p)

$$\int \frac{\ln x}{x}\,dx = \left[\begin{array}{c} t = \ln x \\ dt = \frac{1}{x}dx \end{array}\right] = \int t\,dt = \frac{1}{2}\ln^2|x| + C;$$

(q)

$$\int \frac{\ln^{2005} x}{x}\,dx = \left[\begin{array}{c} t = \ln x \\ dt = \frac{1}{x}dx \end{array}\right] = \int t^{2005}\,dt = \frac{1}{2006}\ln^{2006}|x| + C;$$

(r)

$$\int \frac{1}{x\ln x\ln(\ln x)}\,dx = \left[\begin{array}{c} t = \ln(\ln x) \\ dt = \frac{1}{x\ln x}dx \end{array}\right] = \int \frac{1}{t}\,dt = \ln\Big|\ln\big|\ln|x|\big|\Big| + C;$$

(s)

$$\int \frac{2^{\operatorname{tg} x}}{\cos^2 x}dx = \left[\begin{array}{c} t = \operatorname{tg} x \\ dt = \frac{1}{\cos^2 x}dx \end{array}\right] = \int 2^t\,dt = \frac{1}{\ln 2}2^t + C = \frac{1}{\ln 2}2^{\operatorname{tg} x} + C.$$

4. Koristeći formulu za parcijalnu integraciju izračunati sledeće integrale:

(a) $\int x\sin x\,dx$; (b) $\int (2x-5)\cos x\,dx$; (c) $\int x^2 e^x\,dx$;

(d) $\int (3x^2-5x+6)\,e^{4x-1}\,dx$; (e) $\int x\ln x\,dx$; (f) $\int x^{2005}\ln x\,dx$;

(g) $\int \ln x\,dx$; (h) $\int \operatorname{arctg} x\,dx$; (i) $\int x\operatorname{arctg} x\,dx$;

(j) $\int e^x\sin x\,dx$; (k) $\int \sqrt{1-x^2}\,dx$; (l) $\int \sqrt{1+x^2}\,dx$.

Rešenje:

(a)

$$\int x\sin x\,dx = \left[\begin{array}{c} u = x,\ du = dx \\ dv = \sin x\,dx,\ v = -\cos x \end{array}\right] = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C;$$

(b)

$$\int (2x-5)\cos x\,dx = \left[\begin{array}{c} u = 2x-5,\ du = 2\,dx \\ dv = \cos x\,dx,\ v = \sin x \end{array}\right]$$

$$= (2x-5)\sin x - 2\int \sin x\,dx = (2x-5)\sin x + 2\cos x + C;$$

(c)

$$\int x^2 e^x\,dx = \left[\begin{array}{c} u = x^2,\ du = 2x\,dx \\ dv = e^x\,dx,\ v = e^x \end{array}\right]$$

$$= x^2 e^x - 2\int x e^x\,dx = \left[\begin{array}{c} u = x,\ du = dx \\ dv = e^x\,dx,\ v = e^x \end{array}\right]$$

$$= x^2 e^x - 2(xe^x - \int e^x\,dx) = x^2 e^x - 2xe^x + 2e^x + C;$$

(d)

$$\int (3x^2-5x+6)\,e^{4x-1}\,dx = \left[\begin{array}{c} u = 3x^2-5x+6,\ du = (6x-5)\,dx \\ dv = e^{4x-1}\,dx,\ v = \frac{1}{4}\,e^{4x-1} \end{array}\right]$$

$$= \frac{1}{4}(3x^2-5x+6)\,e^{4x-1} - \frac{1}{4}\int (6x-5)\,e^{4x-1}\,dx$$

$$= \left[\begin{array}{c} u = 6x-5,\ du = 6\,dx \\ dv = e^{4x-1}\,dx,\ v = \frac{1}{4}\,e^{4x-1} \end{array}\right] = \frac{1}{4}(3x^2-5x+6)\,e^{4x-1}$$

$$-\frac{1}{4}\Big[\frac{1}{4}(6x-5)\,e^{4x-1} - \frac{3}{2}\int e^{4x-1}\,dx\Big] = (\frac{3}{4}x^2 - \frac{13}{8}x + \frac{61}{32})\,e^{4x-1} + C;$$

(e)

$$\int x\ln x\,dx = \left[\begin{array}{c} u = \ln x,\ du = \frac{1}{x}\,dx \\ dv = x\,dx,\ v = \frac{x^2}{2} \end{array}\right] = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\,\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C;$$

(f)

$$\int x^{2005}\ln x dx = \left[\begin{array}{c} u=\ln x,\, du=\frac{1}{x}dx \\ dv=x^{2005}dx, v=\frac{x^{2006}}{2006} \end{array}\right] = \frac{x^{2006}}{2006}\ln x - \int \frac{x^{2006}}{2006}\frac{1}{x}dx$$

$$= \frac{x^{2006}}{2006}\ln x - \frac{1}{2006}\int x^{2005}\,dx = \frac{x^{2006}}{2006}\ln x - \frac{x^{2006}}{(2006)^2} + C;$$

(g)

$$\int \ln x\, dx = \left[\begin{array}{c} u=\ln x,\ du=\frac{1}{x}\,dx \\ dv=dx,\ v=x \end{array}\right] = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x + C;$$

(h)

$$\int \operatorname{arctg} x\, dx = \left[\begin{array}{c} u=\operatorname{arctg} x, du=\frac{1}{1+x^2}dx \\ dv=dx, v=x \end{array}\right] = x\operatorname{arctg} x - \int \frac{x}{1+x^2}\,dx$$

$$= \left[\begin{array}{c} t=1+x^2 \\ dt=2xdx \end{array}\right] = x\operatorname{arctg} x - \frac{1}{2}\int \frac{1}{t}\,dt = x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2)+C;$$

(i)

$$\int x\operatorname{arctg} x\, dx = \left[\begin{array}{c} u=\operatorname{arctg} x,\ du=\frac{1}{1+x^2}\,dx \\ dv=x\,dx,\ v=\frac{x^2}{2} \end{array}\right]$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \left(1-\frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{x}{2} + \frac{1}{2}\operatorname{arctg} x + C;$$

(j)

$$\int e^x \sin x\, dx = \left[\begin{array}{c} u=e^x,\ du=e^x\,dx \\ dv=\sin x\,dx,\ v=-\cos x \end{array}\right]$$

$$= -e^x\cos x + \int e^x \cos x\, dx = \left[\begin{array}{c} u=e^x,\ du=e^x\,dx \\ dv=\cos x\,dx,\ v=\sin x \end{array}\right]$$

$$= -e^x\cos x + e^x\sin x - \int e^x \sin x\, dx.$$

Ako uporedimo početni integral i poslednji izraz, imamo da je

$$\int e^x \sin x\, dx = (\sin x - \cos x)\, e^x - \int e^x \sin x\, dx,$$

što ako integral s desne strane prebacimo na levu postaje

$$2\int e^x \sin x\,dx = (\sin x - \cos x)\,e^x,$$

odakle dobijamo da je konačno rešenje:

$$\int e^x \sin x\,dx = \frac{1}{2}(\sin x - \cos x)\,e^x + C;$$

(k) Uradimo prvo metodom smene naredni integral:

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} t = 1-x^2 \\ dt = -2x\,dx \end{bmatrix} = -\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = -\sqrt{1-x^2} + C.$$

Izračunati integral upotrebićemo u narednom radu:

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int x\,\frac{x}{\sqrt{1-x^2}}\,dx \\ &= \begin{bmatrix} u = x,\ du = dx \\ dv = \frac{x}{\sqrt{1-x^2}}\,dx,\ v = -\sqrt{1-x^2} \end{bmatrix} \\ &= \arcsin x + x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx.\end{aligned}$$

Uporedimo li integral koji želimo da rešimo sa poslednjim izrazom imamo da je:

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C;$$

(l) Imamo da je

$$\begin{aligned}\int \sqrt{1+x^2}\,dx &= \int \frac{1}{\sqrt{1+x^2}}\,dx + \int x\,\frac{x}{\sqrt{1+x^2}}\,dx \\ &= \begin{bmatrix} u = x,\ du = dx \\ dv = \frac{x}{\sqrt{1+x^2}}\,dx,\ v = \sqrt{1+x^2} \end{bmatrix} \\ &= \ln|x + \sqrt{1+x^2}| + x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,dx.\end{aligned}$$

Uporedimo li početni i krajnji izraz imamo da je

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\Big(\ln|x + \sqrt{1+x^2}| + x\sqrt{1+x^2}\Big) + C.$$

6. Rešiti sledeće integrale trigonometrijskih funkcija:

(a) $\int \operatorname{tg} x\,dx$; (b) $\int \sin^3 x \cos x\,dx$;

(c) $\int \cos^4 x \sin^3 x\,dx$; (d) $\int \sin^2 x\,dx$; (e) $\int \cos^2 x\,dx$;

(f) $\int \sin^3 x\,dx$; (g) $\int \cos^3 x\,dx$; (h) $\int \sin^4 x\,dx$;

(i) $\int \cos^4 x\,dx$; (j) $\int \sin^2 x \cos^2 x\,dx$;

(k) $\int \operatorname{tg}^2 x\,dx$; (l) $\int \sqrt{1-x^2}\,dx$; (m) $\int \sqrt{4-x^2}\,dx$.

Rešenje:

(a)

$$\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x\,dx \end{bmatrix} = -\int \frac{1}{t}\,dt$$
$$= -\ln|t| + C = -\ln|\cos x| + C;$$

(b)

$$\int \sin^3 x \cos x\,dx = \begin{bmatrix} t = \sin x \\ dt = \cos x\,dx \end{bmatrix} = \int t^3\,dt = \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C;$$

(c)

$$\int \cos^4 x \sin^3 x\,dx = \int \cos^4 x \sin^2 x \sin x\,dx$$
$$= \int \cos^4 x\,(1-\cos^2 x)\,\sin x\,dx = \int (\cos^4 x - \cos^6 x)\,\sin x\,dx$$
$$= \begin{bmatrix} t = \cos x \\ dt = -\sin x\,dx \end{bmatrix} = -\int (t^4 - t^6)\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C$$
$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C;$$

(d)

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \int \frac{1}{2}\,dx - \int \frac{\cos(2x)}{2}\,dx$$
$$= \begin{bmatrix} t = 2x \\ dt = 2dx \end{bmatrix} = \frac{1}{2}\int dx - \frac{1}{2}\int \cos t\,\frac{dt}{2} = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C;$$

(e)

$$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C;$$

(f)

$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x) \sin x\,dx$$

$$= \int \sin x\,dx - \int \cos^2 x \sin x\,dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x\,dx \end{bmatrix}$$

$$= -\cos x + \int t^2\,dt = -\cos x + \frac{\cos^3 x}{3} + C;$$

(g)

$$\int \cos^3 x\,dx = \int \cos x\,dx - \int \sin^2 x \cos x\,dx = \sin x - \frac{\sin^3 x}{3} + C;$$

(h)

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \int \Big(\frac{1-\cos(2x)}{2}\Big)^2 dx = \frac{1}{4}\int dx$$

$$-\frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx = \frac{1}{4}x - \frac{1}{4}\sin(2x)$$

$$+\frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C;$$

(i)

$$\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C;$$

(j) *I način.*

$$\int \sin^2 x \cos^2 x\,dx = \int \sin^2 x(1-\sin^2 x)\,dx = \int \sin^2 x\,dx - \int \sin^4 x\,dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C;$$

II način.

$$\int \sin^2 x \cos^2 x\,dx = \int \frac{1}{4}\sin^2(2x)\,dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C;$$

(k)

$$\int \mathrm{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int dx$$
$$= \mathrm{tg}\,x - x + C;$$

(l)

$$\int \sqrt{1-x^2}\,dx = \left[\begin{array}{c} x = \sin t \\ dx = \cos t\,dt \end{array}\right] = \int \sqrt{1-\sin^2 t}\,\cos t\,dt$$
$$= \int \cos^2 t\,dt = \frac{1}{2}t + \frac{1}{4}\sin(2t) + C = \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + C;$$

(m)

$$\int \sqrt{4-x^2}\,dx = \left[\begin{array}{c} x = 2\sin t \\ dx = 2\cos t\,dt \end{array}\right] = 2\arcsin\frac{x}{2} + \sin(2\arcsin\frac{x}{2}) + C.$$

7. Rešiti sledeće integrale trigonometrijskih racionalnih funkcija:
   (a) $\int \frac{dx}{\cos x}$; (b) $\int \frac{\sin x}{2+\cos x}\,dx$;
   (c) $\int \frac{\sin x + \cos x}{1+\cos x}\,dx$; (d) $\int \frac{\cos x}{\sin x + \cos x \sin x}\,dx$.

   Rešenje:

   (a) Koristeći smenu $t = \mathrm{tg}\frac{x}{2}$ dobijamo:

$$\int \frac{dx}{\cos x} = \int \frac{\frac{2}{1+t^2}}{\frac{1-t^2}{1+t^2}}\,dt = \int \frac{2}{1-t^2}\,dt.$$

   Vidimo da smo početni integral datom smenom sveli na integral racionalne funkcije, pa dalje rešavamo na već poznat način:

$$\int \frac{2}{1-t^2}\,dt = \int \frac{A}{1-t}\,dt + \int \frac{B}{1+t}\,dt.$$

   Rešavajući sistem:

$$\begin{array}{rcl} A+B &=& 2 \\ A-B &=& 0 \end{array}$$

   izračunavamo da je $A = B = 1$, pa je početni integral

$$\int \frac{dx}{\cos x} = -\ln|1-t| + \ln|1+t| + C = \ln\left|\frac{1+tg\frac{x}{2}}{1-tg\frac{x}{2}}\right| + C;$$

(b) Imamo da je

$$\int \frac{\sin x}{2+\cos x}\,dx = [\mathrm{tg}\,\frac{x}{2}=t] = \int \frac{\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt$$

$$= \int \frac{4t}{(3+t^2)(1+t^2)}\,dt = 4\Big(\int \frac{At+B}{3+t^2}\,dt + \int \frac{Ct+D}{1+t^2}\,dt\Big),$$

odakle rešavajući sistem dobijamo da je $A=-\frac{1}{2}, B=0, C=\frac{1}{2}, D=0,$ pa je

$$\int \frac{\sin x}{2+\cos x}\,dx = -2\int \frac{t}{3+t^2}\,dt + 2\int \frac{t}{1+t^2}\,dt$$

$$= \ln\frac{1+t^2}{3+t^2} + C = \ln\frac{1+tg^2\frac{x}{2}}{3+tg^2\frac{x}{2}} + C;$$

(c) Izvršimo li poznatu smenu, imamo da je

$$\int \frac{\sin x+\cos x}{1+\cos x}\,dx = \int \frac{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = -\int \frac{t^2-2t-2}{t^2+1}\,dt.$$

Polinom u brojiocu i imeniocu razlomka su istog stepena, pa nakon deljenja dobijamo:

$$-\int \frac{t^2-2t-2}{t^2+1}\,dt = -\int dt + 2\int \frac{t+1}{t^2+1}\,dt = -t + \int \frac{2t}{t^2+1}\,dt$$

$$+2\int \frac{dt}{1+t^2} = -t + \ln(1+t^2) + 2\ \mathrm{arctgt}\ + C$$

$$= -\mathrm{tg}\,\frac{x}{2} + \ln(1+\ \mathrm{tg}^2\frac{x}{2}) + x + C;$$

(d)

$$\int \frac{\cos x}{\sin x+\cos x\sin x}\,dx = \int \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}\cdot(1+\frac{1-t^2}{1+t^2})}\cdot\frac{2}{1+t^2}\,dt = \int \frac{1-t^2}{2t}\,dt$$

$$= \frac{1}{2}\ln|t| - \frac{t^2}{4} + C = \frac{1}{2}\ln|tg\frac{x}{2}| - \frac{1}{4}\cdot tg^2\frac{x}{2} + C.$$

8. Koristeći metod Ostrogradskog izračunati sledeće integrale:

(a) $\displaystyle\int \frac{3x^3+5}{\sqrt{x^2+4}}\,dx$; (b) $\displaystyle\int \frac{x^2+5x+6}{\sqrt{x^2+2x-3}}\,dx$; (c) $\displaystyle\int \frac{2x^2-6}{\sqrt{x^2+2x+2}}\,dx$.